
Conformal Field Theory and Gravity

Solutions to Problem Set 11

Fall 2024

1. BTZ black hole

Lorentzian AdS_3 (with closed timelike curves) is defined as

$$-(X^0)^2 + (X^1)^2 + (X^2)^2 - (X^3)^2 = -\ell^2 \quad (1)$$

which has the global solution

$$\begin{pmatrix} X^0 \\ X^3 \end{pmatrix} = \ell \cosh \lambda \begin{pmatrix} \cos(t/\ell) \\ \sin(t/\ell) \end{pmatrix}, \quad \begin{pmatrix} X^1 \\ X^2 \end{pmatrix} = \ell \sinh \lambda \begin{pmatrix} \cos(t/\ell) \\ \sin(t/\ell) \end{pmatrix}, \quad (2)$$

with $\lambda \in \mathbb{R}$, $\varphi \sim \varphi + 2\pi$, and $t \sim t + 2\pi\ell$ (this represents the closed timelike curve). The metric reads

$$ds^2 = -\cosh^2(\lambda) dt^2 + \ell^2 d\lambda^2 + \ell^2 \sinh^2(\lambda) d\varphi^2 \quad (3)$$

and at this point we can “unroll” the time coordinate t , taking $t \in \mathbb{R}$. After passing to a new coordinate $r \in \mathbb{R}$ via

$$\sinh \lambda = \frac{r}{\ell}, \quad (4)$$

the projected metric reads

$$ds^2 = -\tilde{f}(r) dt^2 + \frac{dr^2}{\tilde{f}(r)} + r^2 d\varphi^2, \quad \tilde{f}(r) = 1 + \frac{r^2}{\ell^2}. \quad (5)$$

(a) The metric of the exercise is very similar to (5), except we have

$$f_{\text{BTZ}}(r) = f(-8GM; r), \quad f(c; r) := c + \frac{r^2}{\ell^2} \quad (6)$$

instead of $\tilde{f}(r)$. Notice that GM is dimensionless, since by its definition the mass dimension of G is $[G] = 2 - d = -1$ in $2 + 1$ spacetime dimensions. In fact, we can study the curvature tensors of the metric (34) keeping c arbitrary. Then for any c , we have the identity¹

$$R_{\mu\nu\rho\sigma} = -\frac{1}{\ell^2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \quad (7)$$

so locally, $g_{\mu\nu}$ is maximally symmetric, with scalar curvature $R = -6/\ell^2$. Since this holds for any c , it holds in particular for f_{BTZ} .

(b) Since the metric only depends on r , it's easy to see that $\xi_{(1)} = \partial_t$ and $\xi_{(2)} = \partial_\varphi$ are Killing vectors. The global AdS_3 metric, with $c = 1$, has 6 Killing vectors, since that is the dimension of $\text{SO}(2,2)$. However, the 4 remaining ones cease to be solutions of $\mathcal{L}_\xi g_{\mu\nu} = 0$ when $c \neq 1$.²

¹This is easiest to check in Mathematica.

²To fully explore this claim, one should actually write down the equations $\mathcal{L}_\xi g_{\mu\nu} = 0$, which is a set of 6 equations (since $g_{\mu\nu}$ has 6 independent components) for 3 functions $(\xi^t, \xi^r, \xi^\varphi)$. This is a tedious exercise.

(c) The norm of the timelike Killing vector is

$$-\xi_{(1)} \cdot \xi_{(1)} = f_{\text{BTZ}}(r) = \frac{r^2 - r_h^2}{\ell^2}. \quad (8)$$

This is positive (resp. negative) if $r > r_h$ (resp. $r < r_h$), so $r = r_h$ is indeed a Killing horizon.

(d) Let's use the Brown-York formalism. Given a Killing vector ξ^a (pushed to the boundary), the corresponding conserved charge is

$$Q[\xi] = \lim_{r \rightarrow \infty} \int_{\partial \Sigma} d^{d-2} x \sqrt{\sigma} n^a T_{ab} \xi^b = \lim_{r \rightarrow \infty} \frac{1}{8\pi G} \int_0^{2\pi} d\varphi \sqrt{\gamma} \Delta K_{ab} n^a \xi^b \quad (9)$$

with

$$\Delta K_{ab} := K_{ab} - \gamma_{ab} K - \frac{1}{\ell} \gamma_{ab}. \quad (10)$$

Here K_{ab} is the 2×2 projection of the exterior curvature $K_{\mu\nu} = -\nabla_\nu(\sigma\nu_\mu)$, the vector field $\sigma^\mu \partial_\mu = \sqrt{f_{\text{BTZ}}(r)} \partial_r$ is the unit-normalized outward facing normal, and γ_{ab} is the 2×2 metric of the boundary of AdS. Finally n^a is the projection of n^μ , the unit normal orthogonal to the timeslice Σ .

A straightforward computation (keeping c arbitrary for now) gives

$$K_{\mu\nu}(r) = r \sqrt{f(r)} \times \text{diag} \left(\frac{1}{\ell^2}, 0, -1 \right). \quad (11)$$

This is indeed of rank 2, since $K^{\mu\nu} = 0$. Projecting to the boundary coordinates $x^a = \{t, \varphi\}$ gives the non-degenerate 2×2 matrix

$$\Delta K_{ab} = -\frac{c}{2} \times \text{diag} \left(\frac{1}{\ell}, \ell \right) + O(1/r^2) \quad (12)$$

with individual components

$$\Delta K_{tt} \approx \frac{4GM}{\ell}, \quad \Delta K_{\varphi\varphi} \approx 4GM\ell, \quad \Delta K_{t\varphi} = 0. \quad (13)$$

Setting c to its BTZ value $c = -8GM$, and discarding $O(1/r^2)$ terms. In this formula, the subtraction term $(1/\ell) \times \gamma_{ab}$ is crucial — without it, terms proportional to r appear that diverge near the boundary of AdS.

Let's inspect the large- r behavior of the BY integrand (38) in more detail. The (d-2)-dimensional boundary metric is $\sigma_{\varphi\varphi} = r^2$, so $\sqrt{\sigma} = r$. Yet $n^\mu = \frac{1}{\sqrt{f_{\text{BTZ}}(r)}}(1, 0, 0)$, so after projecting to the boundary, we have $n^a \sim (\ell/r, 0)$ as $r \rightarrow \infty$. It follows that $\sqrt{\sigma} n^a \sim (\ell, 0)$ as $r \rightarrow \infty$. Thus, if we specialize to the timelike Killing vector $\xi^a = (1, 0)$, we get

$$E = \lim_{r \rightarrow \infty} \frac{1}{8\pi G} \int_0^{2\pi} d\varphi \sqrt{\sigma} n^t \Delta K_{tt} + O(1/r) = \frac{1}{8\pi G} \cdot 2\pi\ell \cdot \frac{4GM}{\ell} \quad (14)$$

$$= M. \quad (41)$$

Note that this is a dimensionful result; we could have defined the conserved charge with an additional factor of ℓ to make it dimensionless.

In passing, you can easily show that the spin J of the BTZ black hole from the exercise vanishes, since $\Delta K_{t\varphi} = 0$.

(e) Intrinsically, the surface gravity κ is defined as

$$\xi^\mu \nabla_\mu \xi^\nu = \kappa \xi^\nu \quad (42)$$

which is to be evaluated at the Killing horizon. This is awkward to compute in the metric (5), since $g_{rr} \rightarrow \infty$ precisely at $r = r_h$. Either we can change coordinates to something akin to Kruskal coordinates, or we use the formula

$$\kappa^2 = -\frac{1}{2} \nabla^\mu \xi^\nu \nabla_\mu \xi_\nu \Big|_{r=r_h}, \quad (43)$$

which works in any coordinate system (for a derivation, see Wald, Eq. (12.5.14)).

What remains is to compute $\nabla_\mu \xi_\nu$ for the Killing vector $\xi^\mu \partial_\mu = \partial_t$. This is straightforward, for example in Mathematica. We find that

$$\nabla_r \xi_t = -\frac{r}{\ell^2}, \quad \nabla_t \xi_r = \frac{r}{\ell^2}, \quad (44)$$

and $\nabla_\mu \xi_j = 0$ for all other components. (In particular, $\nabla_\mu \xi^\mu + \nabla_\nu \xi^\nu = 0$, as befits a Killing vector.) Moreover, $g^{tt} g_{rr} = -1$, so $\nabla^\mu \xi^\nu = -\nabla_\mu \xi_\nu$ numerically. It follows that

$$\kappa^2 = -\frac{1}{2}(-1) \cdot 2 \left[\frac{r_h}{\ell^2} \right]^2 \Rightarrow \kappa = \frac{r_h}{\ell^2} = \sqrt{\frac{8GM}{\ell}}. \quad (45)$$

(f) The “area” A of a black hole in $2 + 1$ d is its circumference.³ For the BTZ black hole, this is $A = 2\pi r_h$. Hence the Bekenstein-Hawking entropy reads

$$S_{\text{BH}} = \frac{A}{4G} = \frac{\pi r_h}{2G} = \sqrt{\frac{2M}{G}} \pi \ell. \quad (46)$$

(g) The Euclidean version of the BTZ geometry is

$$ds^2 = f(r) dr^2 + \frac{dr^2}{f(r)} + r^2 d\varphi^2. \quad (47)$$

where $\tau \sim \tau + \beta$ for some inverse temperature $\beta > 0$. Just as before, you can check that locally, this is a maximally symmetric space with $R = -6/\ell^2$ for any c , and in particular for $c = -8GM$. Now, the function $f_{\text{BTZ}}(r)$ has a root at $r = r_h$. By the conical defect trick, we must set

$$\beta = \frac{4\pi}{f'(r_h)} \Rightarrow T = \frac{1}{\beta} = \frac{\sqrt{2GM}}{\pi \ell}.$$

This computes the temperature of the BTZ black hole. Finally, since $T = \kappa/(2\pi)$, we predict that

$$\kappa = 2\pi T = \frac{\sqrt{8GM}}{\ell}.$$

This matches the previous result.

³A black hole in a spacetime of total dimension $D = d + 1$ has a surface area $A \sim R^{d-1}$ for some length scale R . At the same time, in d dimensions, the Newton constant has mass dimension $[G] = 2 - D = 1 - d$. Therefore $S = A/(4G)$ is dimensionless in any dimension, as befits an entropy.

- (h) The central charge in AdS_3 is related to the AdS radius via

$$c = \frac{3\ell}{2G}.$$

We can now apply the Cardy formula with $E = M\ell$, which gives

$$S_{\text{Cardy}} = \sqrt{\frac{2M}{G}} \pi \ell.$$

This matches precisely with the Bekenstein-Hawking entropy S_{BH} that you computed previously.

2. The Cardy formula

- (a) We interpret $Z(\beta, R)$ as the partition function on a torus which has circumferences β and $2\pi R$. Of course we can swap the circumferences $\beta \leftrightarrow 2\pi R$ (corresponding to swapping one space with one (Euclidean) time direction) as pointed out in the main text. So that we have a torus with circumferences $2\pi R$ and β . (and now β is interpreted as the circumference of the circle on which the QFT lives, with radius $\beta/(2\pi)$, where as $2\pi R$ is the length of the compactified “time” direction.).

In a CFT, we also have scale invariance. Thus, we can rescale

$$(2\pi R, \beta) \rightarrow 2\pi \frac{R}{\beta} \cdot (2\pi R, \beta) = \left(\frac{4\pi^2 R^2}{\beta}, 2\pi R \right) \quad (15)$$

so we have a torus with circumferences $4\pi^2 R^2/\beta$ and $2\pi R$. This is interpreted as a CFT again on a spatial circle of radius R but now the time direction is compactified on a circumference $\sim 1/\beta$,

$$Z(\beta, R) = Z\left(\frac{4\pi^2 R^2}{\beta}, R\right) \quad (16)$$

This is interesting because it relates the infinite temperature limit to the low temperature limit.

- (b) The easiest limit is the low temperature limit $\beta \rightarrow \infty$ where $Z = \sum_n e^{-\beta E_n}$ is dominated by the state n which lowest energy. For a unitary 2d CFT, this state has the Casimir energy $E_0 = -\frac{c}{12R}$. Thus,

$$Z(\beta, R) \approx e^{\frac{\beta c}{12R}} \quad (\beta \rightarrow \infty) \quad (17)$$

Using (16), this gives the infinite temperature $\beta \rightarrow 0$ behaviour as well

$$Z(\beta, R) \approx e^{\frac{4\pi^2 R}{\beta} \frac{c}{12}} \quad (\beta \rightarrow 0) \quad (18)$$

To obtain the entropy, we compute the energy density of states defined through

$$Z(\beta) = \int \rho(E) e^{-\beta E} dE \quad (19)$$

Writing $\rho(E) = e^{S(E)}$ where S is the entropy, we want to solve S such that it gives the behaviour (18). It is useful to rescale $E = \hat{E}/\beta$ so that

$$Z(\beta) = \frac{1}{\beta} \int e^{S(\hat{E}/\beta) - \hat{E}} d\hat{E} \quad (20)$$

We now perform a saddle point approximation in the large $1/\beta$ limit. The integral will be dominated around the saddle for which

$$\frac{d}{d\hat{E}}(S(\hat{E}/\beta) - \hat{E}) = 0 \quad (21)$$

It is not hard to convince oneself that $S(E) \sim \sqrt{E}$ is the correct choice to reproduce (18). Indeed, for such a choice the saddle is at $\hat{E} \sim 1/\beta$ so that it contributes $\sim e^{1/\beta}$. Fixing the factors, we obtain

$$S(E) = 2\pi \sqrt{\frac{cER}{3}} \quad (22)$$

3. Asymptotic charges in gauge theory

- (a) The infinitesimal variation of the fields under this transformation is:

$$\delta\phi = i\epsilon\phi, \quad \delta\phi^* = -i\epsilon\phi^*.$$

The conserved current is:

$$j_\phi^\mu = i(\partial^\mu\phi^*\phi - \phi^*\partial^\mu\phi).$$

The conserved charge associated with this current on a spacelike hypersurface Σ is:

$$Q_1 = \int_\Sigma d^{D-1}x j_0 = \int_\Sigma d^{D-1}x i(\partial_0\phi^*\phi - \phi^*\partial_0\phi).$$

- (b) Given any Lagrangian L depending on a set of fields and their derivatives $\{\phi^a, \partial_\mu\phi^a\}$, the action variation can be written as

$$\begin{aligned} \delta S &= \int_M \sum_a \frac{\delta L}{\delta\phi^a} \delta\phi^a + \frac{\delta L}{\delta(\partial_\mu\phi^a)} \partial_\mu\delta\phi^a \\ &= \int_M \sum_a \left(\frac{\delta L}{\delta\phi^a} - \partial_\mu \frac{\delta L}{\delta(\partial_\mu\phi^a)} \right) \delta\phi^a + \partial_\mu \left(\frac{\delta L}{\delta(\partial_\mu\phi^a)} \delta\phi^a \right) \\ &= \int_M EOM + \partial_\mu S^\mu \end{aligned} \quad (23)$$

- (c) From the previous equation, applied to our system, we get (up to an overall minus sign):

$$\begin{aligned} S^\mu &= \frac{\delta L}{\delta(\partial_\mu\phi)} \delta\phi + \frac{\delta L}{\delta(\partial_\mu\phi^*)} \delta\phi^* + \frac{\delta L}{\delta(\partial_\mu A_\nu)} \delta A_\nu \\ &= g\epsilon(x)j_\phi^\mu + (\partial_\nu\epsilon(x))F^{\mu\nu} = \epsilon(x)(gj_\phi^\mu - \partial_\nu F^{\mu\nu}) + \partial_\nu(\epsilon(x)F^{\mu\nu}) \end{aligned} \quad (24)$$

Using the equations of motion,

$$\partial_\nu F^{\mu\nu} = gj_\phi^\mu \quad (25)$$

Then we get

$$S^\mu = \partial_\nu(\epsilon(x)F^{\nu\mu}) \quad (26)$$

- (d) The charge can be found by integrating the current

$$Q_2 = \int_\Sigma S_0 = \int_{\partial\Sigma} \epsilon F_{0i} n^i \quad (27)$$

where n^i is the normal vector to $\partial\Sigma$. This shows that charges in gauge theory can be computed as a boundary integral.

- (e) Charges in gauge theory are 0 if the $U(1)$ transformation is a gauge transformation, i.e. if $\epsilon(x)|_{\partial\Sigma} = 0$. On the other hand, genuine physical symmetries act non-trivially on the boundary and are characterised by $\epsilon(x)|_{\partial\Sigma} \neq 0$, and thus can have a non-zero charge.
- (f) If $g \rightarrow 0$, then the fields ϕ and A^μ decouple, hence the problem is now characterised by two charges: one associated to the $U(1)$ asymptotic symmetries (also known as "large gauge transformations") of the Maxwell field, that are computed as boundary integrals, and one associated to the $U(1)$ global symmetry of the ϕ .